Black Hole Astrophysics
Chapters 7.5

All figures extracted from online sources of from the textbook.

Recap – the Schwarzschild metric

“Sch” means that this metric is describing a Schwarzschild Black Hole.

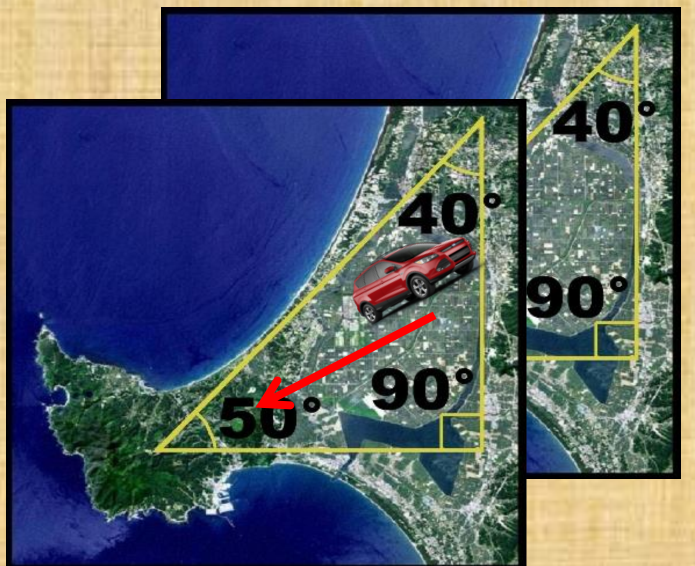
$$\left(g_{\mathrm{SH}}^{\mathrm{Sch}}\right)_{\alpha\beta} = \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The Schwarzschild radius $r_s = \frac{2\mathrm{GM}}{c^2}$

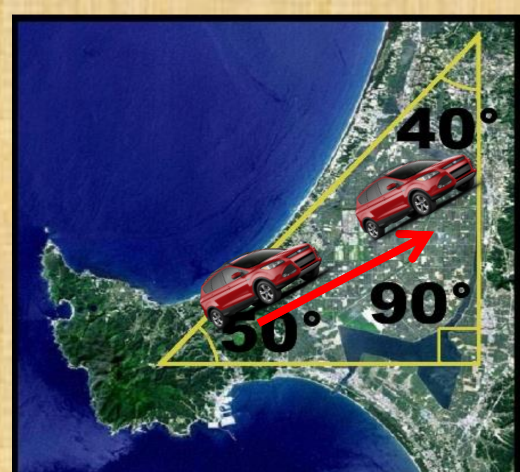
“SH” means that we are in the Schwarzschild-Hilbert coordinate system.

Recap – coordinate systems used

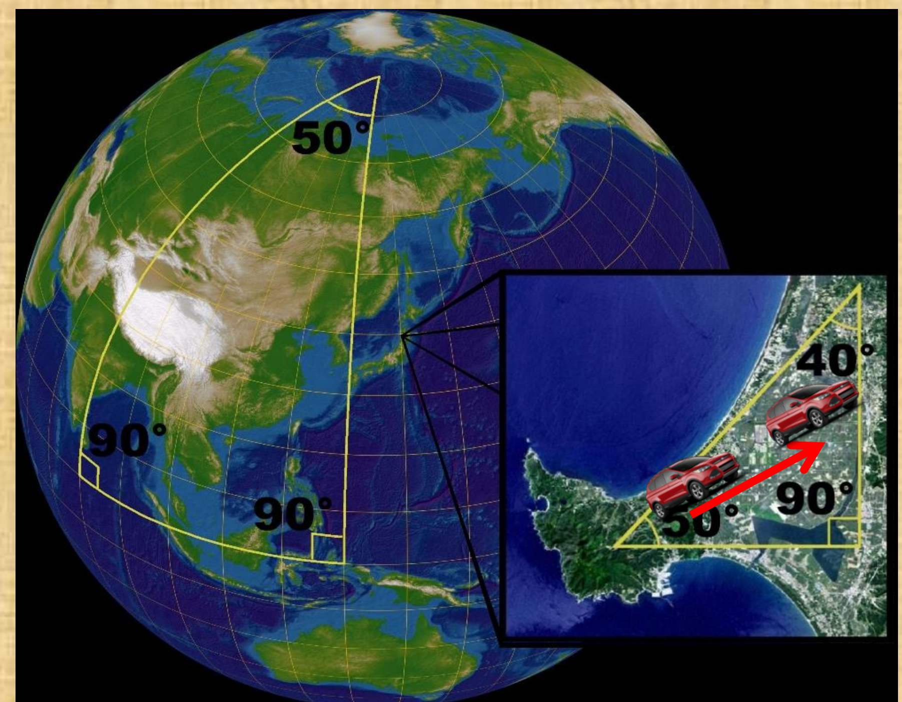
1. The moving body frame (MOV)



2. Fixed local Lorentz frame (FIX)

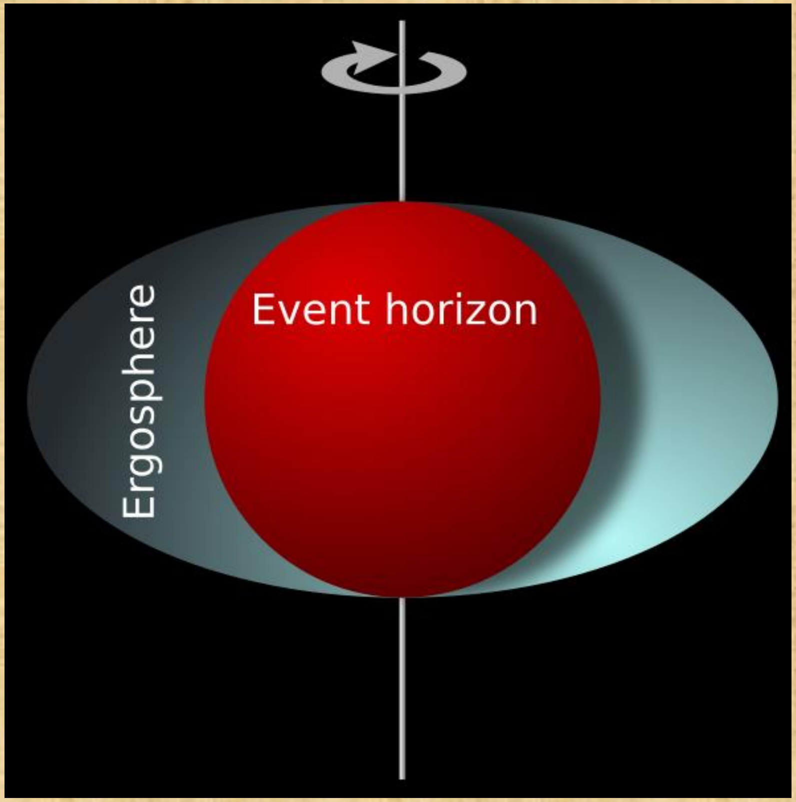


3. Schwarzschild-Hilbert frame (SH)



What is a Kerr Black Hole?

Solution to the Einstein Equations for
Rotating, Uncharged, axially-symmetric black hole in empty spacetime.



Brief Introduction:

Like other stars, black holes can rotate. The difference here, is that black hole rotational speeds can be near the speed of light, causing still more changes in the metric and in the way matter moves near a black hole. Space itself around a black hole can rotate, and it is this rotation of space that, very possibly, is the ultimate cause for some of the powerful jets of matter that we see being ejected from quasars and other black hole systems.

The Kerr Metric

"KER" means that this metric is describing a Kerr Black Hole.

"BL" means that we are in the Boyer-Lindquist coordinate system (generalization of Schwarzschild-Hilbert coordinate).

$$\left(g_{\mathrm{BL}}^{\mathrm{KER}}\right)_{\alpha\beta} = \begin{pmatrix} -\left(1-\frac{2r_g r}{\rho^2}\right)c^2 & 0 & 0 & -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} & 0 & 0 & \frac{\Sigma^2}{\rho^2}\sin^2\theta \end{pmatrix}$$

The Gravitational radius $r_g = \frac{\mathrm{GM}}{c^2}$

Because the metric is sort of complicated, so the 3 functions are defined for clarity.

The Kerr Metric

The Gravitational radius $r_g = \frac{\mathrm{GM}}{c^2}$

$\omega = \frac{2{r_g}^2 rc}{\Sigma^2} j$ angular frequency of space at different points around the BH.

$$\left(g_{\mathrm{BL}}^{\mathrm{KER}}\right)_{\alpha\beta} = \begin{pmatrix} -\left(1 - \frac{2r_g r}{\rho^2}\right)c^2 & 0 & 0 & -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} & 0 & 0 & \frac{\Sigma^2}{\rho^2}\sin^2\theta \end{pmatrix}$$

$j = \frac{J}{GM^2/c}$ is the dimensionless angular momentum parameter ($-1 \leq j \leq 1$) that measures how much angular momentum J the BH has relative to its maximum value.

$\rho \equiv \sqrt{r^2 + j^2 {r_g}^2 \cos^2\theta}$ is the radial distance like r, but constant ρ are oblate; $\lim_{j\to 0} \rho = r$

$\Sigma \equiv \sqrt{\left(r^2 + j^2 {r_g}^2\right)^2 - j^2 {r_g}^2 \Delta \sin^2\theta}$ is the equatorial area and $\frac{\Sigma\sin\theta}{\rho}$ is the cylindrical distance from the rotation axis. $\lim_{j\to 0} \Sigma = r^2$

$\Delta \equiv r^2 - 2r_g r + {r_g}^2 j^2; \lim_{j\to 0} \Delta = r^2 - 2r_g r$

The Horizon and Ergosphere

Previously, with the Schwarzschild BH, both $g_{\mathrm{tt}} \to 0$ and $g_{\mathrm{rr}} \to \infty$at $r = r_s$.

$$\left(g_{\mathrm{SH}}^{\mathrm{Sch}}\right)_{\alpha\beta} = \begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

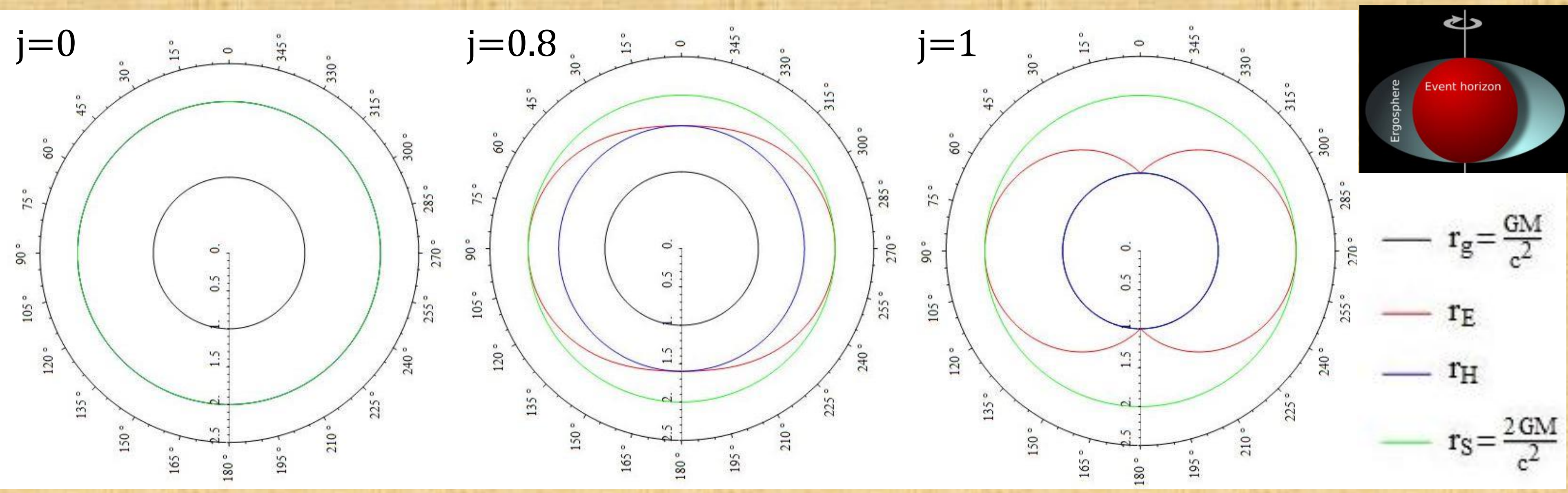
However, for the Kerr metric,

$$\left(g_{\mathrm{BL}}^{\mathrm{KER}}\right)_{\alpha\beta} = \begin{pmatrix} -\left(1-\frac{2r_g r}{\rho^2}\right)c^2 & 0 & 0 & -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} & 0 & 0 & \frac{\Sigma^2}{\rho^2}\sin^2\theta \end{pmatrix}$$

$g_{\mathrm{rr}} \to \infty$ when $\Delta \to 0$ gives $r_H = \left(1+\sqrt{1-j^2}\right)r_g$ which is, in fact, the horizon.

$g_{\mathrm{tt}} \to 0$ when $1-\frac{2r_g r}{\rho^2} = 0$ gives $r_E = \left(1+\sqrt{1-j^2\cos^2\theta}\right)r_g$

We can see that this is larger than r_H except at the poles



The Horizon and Ergosphere

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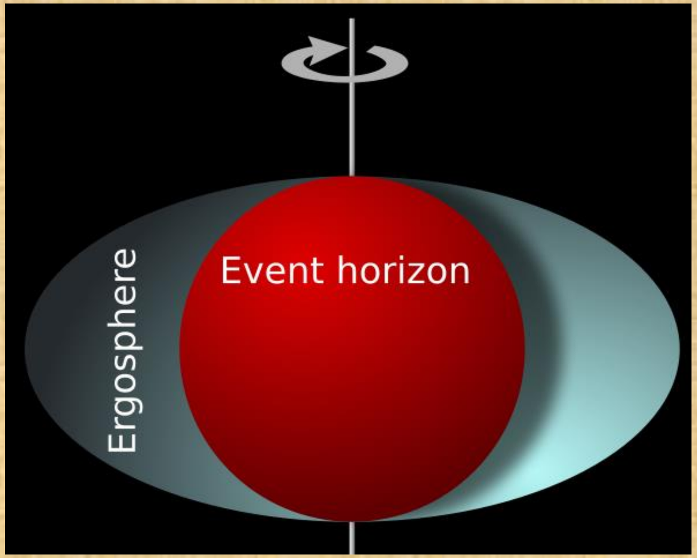


So, what happens within r_E but outside r_H?

From the metric,

$$\left(g_{\mathrm{BL}}^{\mathrm{KER}}\right)_{\alpha\beta} = \begin{pmatrix} -\left(1 - \frac{2 r_g r}{\rho^2}\right) c^2 & 0 & 0 & -\frac{\omega \Sigma^2 \sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{\omega \Sigma^2 \sin^2\theta}{\rho^2} & 0 & 0 & \frac{\Sigma^2}{\rho^2} \sin^2\theta \end{pmatrix}$$

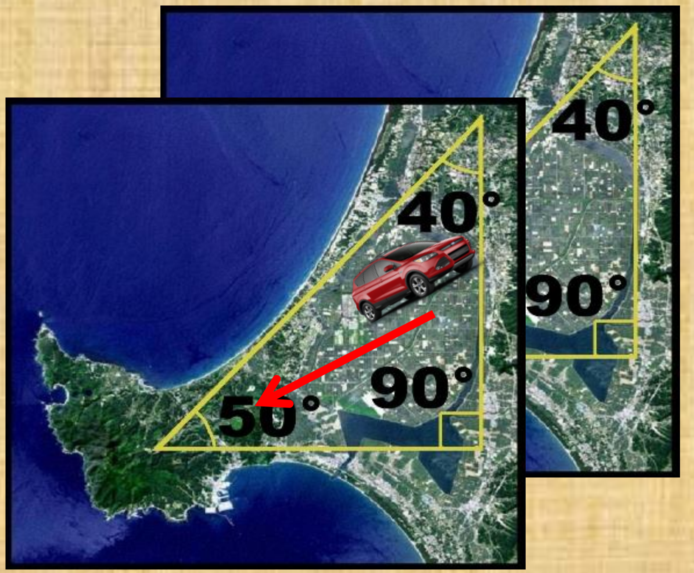
Both g_{tt} and g_{rr} are positive! How will particles behave in this region?

Since particles follow timelike geodesics, $\mathrm{ds}^2 < 0$, this means that we must have the $g_{\mathrm{t\phi}}$ term to be positive. This forces $\mathrm{d}\phi$ to be non-zero!

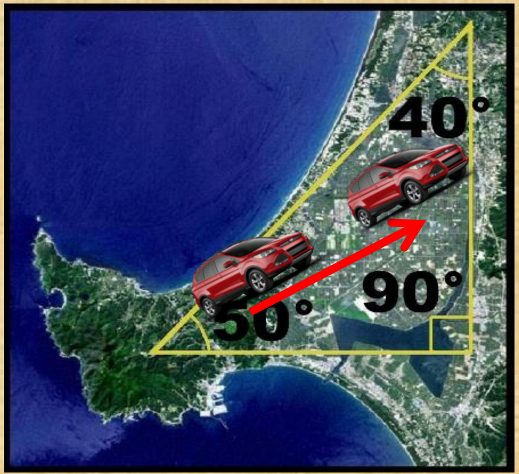
i.e. <u>*In this region, called the "Ergosphere", everything must rotate in the same direction (how to prove?) as the black hole!*</u>

Coordinate systems for Kerr BH

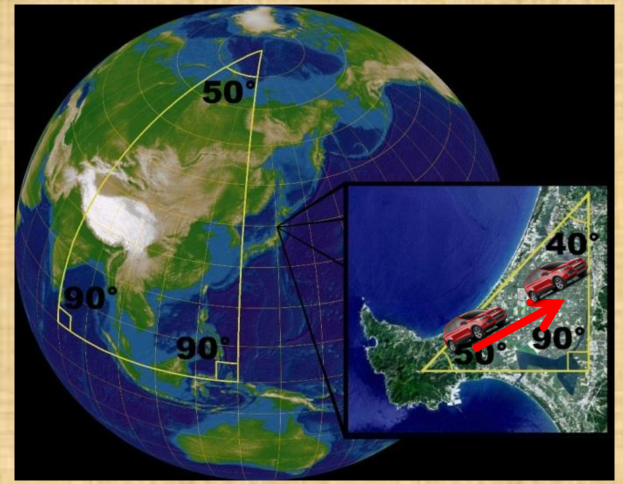
1. The moving body frame (MOV)



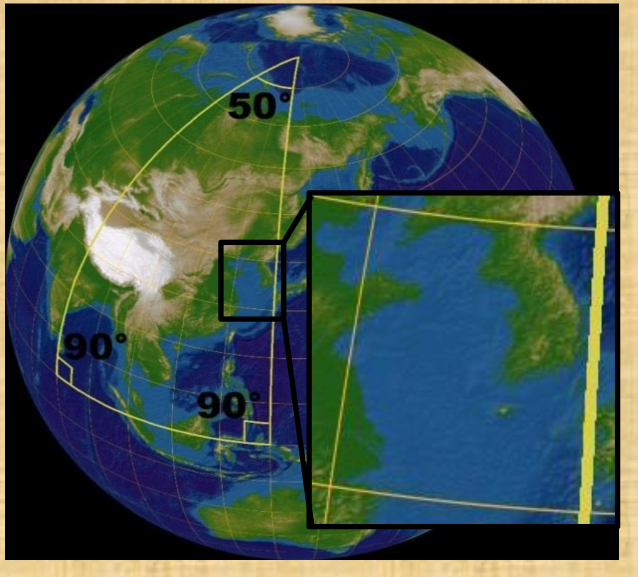
2. Fixed local Lorentz frame (FIX)



3. Boyer-Lindquist frame (BL)



4. Observer at Infinity/Synchronous frame (OIS)



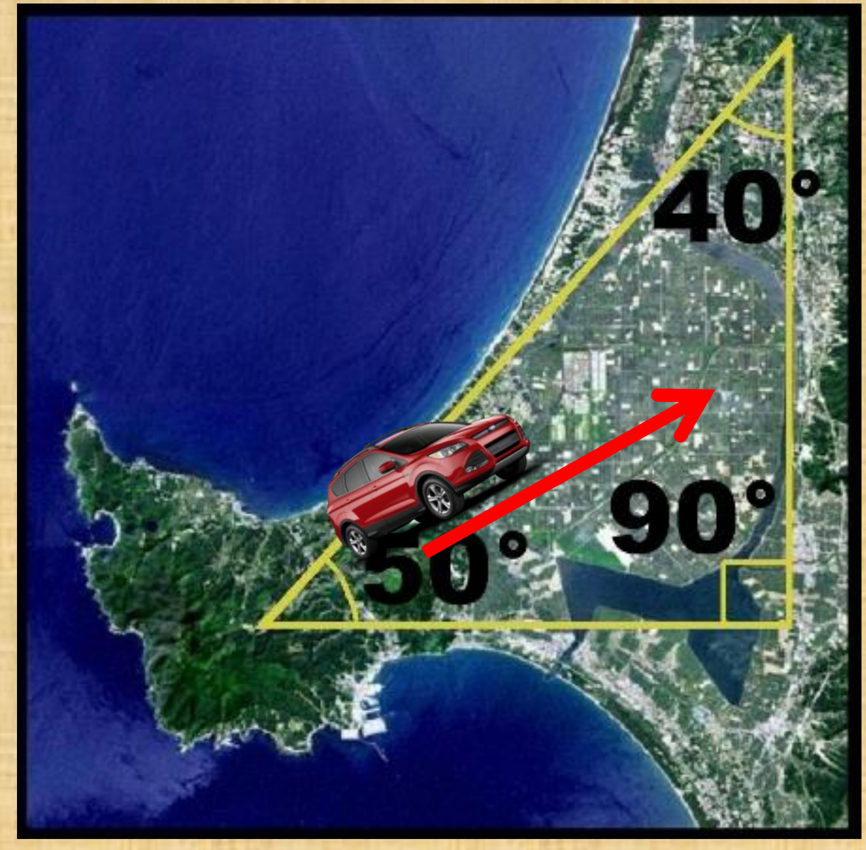
The moving body frame (MOV)

Similar to the Schwarzschild BH case, we can choose to move with the object of interest. Since spacetime is locally flat, we have a Minkowski metric in this case

$$\left(g_{\text{MOV}}^{\text{KER}}\right)_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and by definition the 4-velocity $(U_{\text{MOV}})^{\alpha} = (c, 0,0,0)$

Fixed local Lorentz frame (FIX)



Again, we consider some locally flat part of the Kerr spacetime to sit on and watch things fly past. Therefore the metric is still the Minkowski one

$$\left(g_{\mathrm{FIX}}^{KER}\right)_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

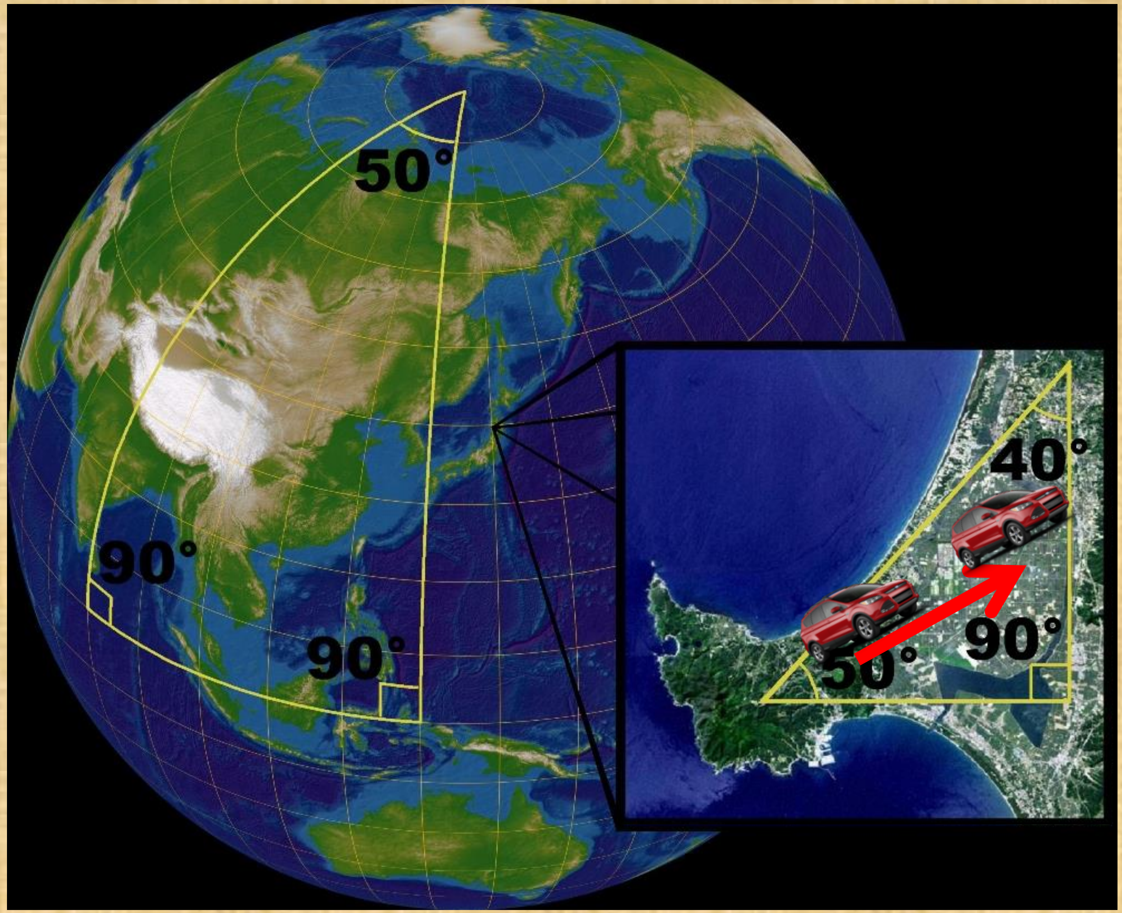
but now the 4-velocity of objects become

$$(U_{\mathrm{FIX}})^{\alpha} = \begin{pmatrix} \gamma c \\ \gamma V^{\hat{r}} \\ \gamma V^{\hat{\theta}} \\ \gamma V^{\hat{\phi}} \end{pmatrix}$$

However, the Kerr spacetime itself is rotating, therefore being fixed in such a frame means that you are actually rotating with the black hole at rate ω.

Therefore this observer has no angular momentum with respect to the BH.

Boyer-Lindquist frame (BL)

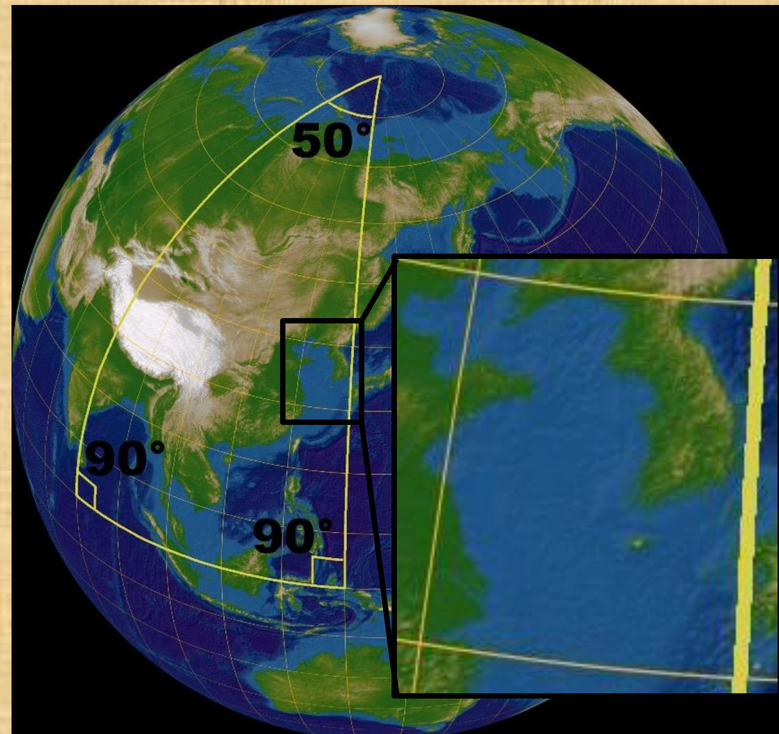


This is a global coordinate, just like the SH frame used for Schwarzschild BH.

For this coordinate, the metric is the one we presented earlier

$$\left(g_{\mathrm{BL}}^{\mathrm{KER}}\right)_{\alpha\beta} = \begin{pmatrix} -\left(1-\dfrac{2r_g r}{\rho^2}\right)c^2 & 0 & 0 & -\dfrac{\omega\Sigma^2\sin^2\theta}{\rho^2} \\ 0 & \dfrac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\dfrac{\omega\Sigma^2\sin^2\theta}{\rho^2} & 0 & 0 & \dfrac{\Sigma^2}{\rho^2}\sin^2\theta \end{pmatrix}$$

Observer at Infinity/Synchronous frame (OIS)



This system rotates with the space around the black hole, is local to each radius r, and has a diagonal metric; but that metric is *not an orthonormal Lorentz/ Minkowskian system . It relates to the BL coordinates simply by* $\mathrm{d}\phi' = \mathrm{d}\phi - \omega \mathrm{d}t$

$$\Lambda_{\mathrm{BL}}^{\mathrm{OIS}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\omega & 0 & 0 & 1 \end{pmatrix}$$

Which give the metric

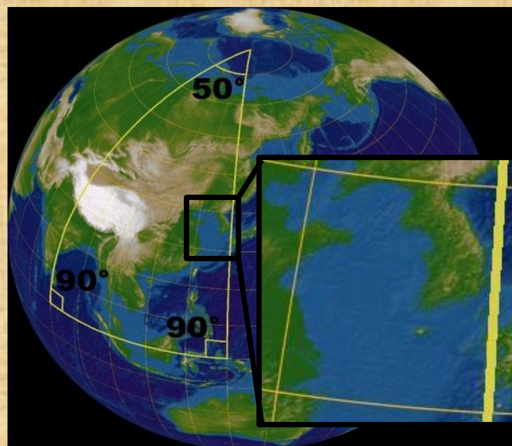
$$(g_{\mathrm{OIS}}^{\mathrm{KER}})_{\alpha\beta} = \begin{pmatrix} -\frac{\rho^2 \Delta}{\Sigma^2} c^2 & 0 & 0 & 0 \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & \frac{\Sigma^2}{\rho^2} \sin^2\theta \end{pmatrix}$$

Following same procedure as last week (see p12 of last week)

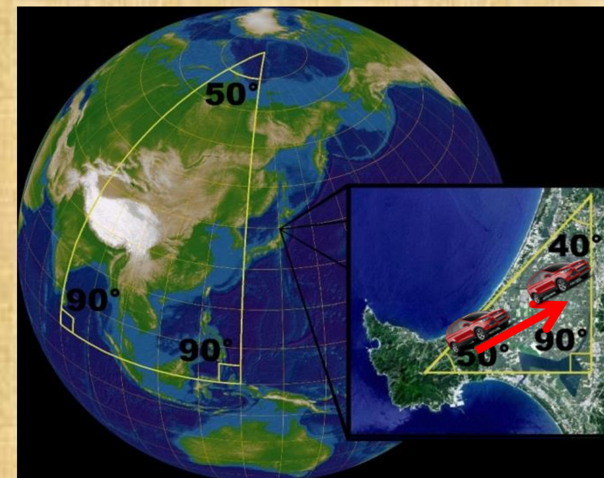
$$(U_{\mathrm{OIS}})^\alpha = \Lambda_{\mathrm{FIX}}^{\mathrm{OIS}} (U_{\mathrm{FIX}})^\alpha = \begin{pmatrix} \frac{\Sigma}{\rho c \sqrt{\Delta}} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{\Delta}}{\rho} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & 0 & \frac{\rho}{\Sigma \sin\theta} \end{pmatrix} \begin{pmatrix} \gamma c \\ \gamma V^{\hat{r}} \\ \gamma V^{\hat{\theta}} \\ \gamma V^{\hat{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma \Sigma / \rho \sqrt{\Delta} \\ \gamma V^{\hat{r}} \sqrt{\Delta} / \rho \\ \gamma V^{\hat{\theta}} / \rho \\ \gamma V^{\hat{\phi}} \rho / \Sigma \sin\theta \end{pmatrix}$$

Back to the Boyer-Lindquist frame

4. Observer at Infinity/ Synchronous frame (OIS)



3. Boyer-Lindquist frame (BL)



The OIS is not a good system in which to work. It is not global, so derivatives are valid only locally. And the ergosphere is hidden in this frame because ω is gone from $g_{\mathrm{OIS}}^{\mathrm{KER}}$. So we quickly leave the OIS system and transform the vectors and 1-forms back to the Kerr system to find

$$(U_{\mathrm{BL}})^{\alpha} = \Lambda_{\mathrm{OIS}}^{\mathrm{BL}} (U_{\mathrm{OIS}})^{\alpha} = \begin{pmatrix} \gamma\Sigma/\rho\sqrt{\Delta} \\ \gamma V^{\hat{r}}\sqrt{\Delta}/\rho \\ \gamma V^{\hat{\theta}}/\rho \\ \dfrac{\gamma V^{\hat{\phi}}\rho}{\Sigma\sin\theta} + \dfrac{\gamma\Sigma\omega}{\rho\sqrt{\Delta}} \end{pmatrix} \qquad (u^{\mathrm{BL}})_{\alpha} = \left(-\left[\gamma c^2 \frac{\rho\sqrt{\Delta}}{\Sigma} + \frac{\gamma V^{\hat{\phi}}\Sigma\sin\theta}{\rho}\,\omega\right], \frac{\gamma\rho V^{\hat{r}}}{\sqrt{\Delta}}, \gamma\rho V^{\hat{\theta}}, \frac{\gamma\Sigma\sin\theta}{\rho} V^{\hat{\phi}} \right)$$

Remember that $\left(V^{\hat{r}}, V^{\hat{\theta}}, V^{\hat{\phi}}\right)$ are all evaluated in the FIX frame.

Conserved Quantities

As we have introduced last week, since the metric is independent of t and ϕ, Energy and angular momentum are conserved.

The Angular momentum:

$$\frac{\mathrm{d}p_\phi}{\mathrm{d}\tau} \equiv \frac{\mathrm{dL}}{\mathrm{d}\tau} = 0$$

$$L = p_\phi^{Kerr} = \gamma m_0 V^{\hat{\phi}} \underline{\frac{\Sigma \sin\theta}{\rho}}$$

$$p_\phi^{\text{Schwarzschild}} = \gamma m_0 V^{\hat{\phi}} \underline{r \sin\theta}$$

Comparing with the Schwarzschild case, it's easy to see that $\frac{\Sigma \sin\theta}{\rho}$ is the cylindrical radius as we mentioned before.

Conserved Quantities

Energy:

$$E^{\text{Schwarzschild}} = -p_t = \underline{\gamma m_0 c^2 \sqrt{1 - \frac{r_s}{r}}}$$

$$\frac{\mathrm{dp}_t}{\mathrm{d}\tau} \equiv -\frac{\mathrm{dE}}{\mathrm{d}\tau} = 0$$

$$E^{\text{Kerr}} = -\mathrm{p}_t = \underline{\gamma m_0 \mathrm{c}^2 \frac{\rho\sqrt{\Delta}}{\Sigma}} + \underline{\gamma m_0 \mathrm{V}^{\hat{\phi}} \frac{\Sigma \sin\theta}{\rho} \omega}$$

~~Energy at infinity~~

$$\gamma m_0 \mathrm{V}^{\hat{\phi}} \frac{\Sigma \sin\theta}{\rho} \omega$$

Is the new term in Kerr metric due to rotation of space.

If $\mathrm{V}^{\hat{\phi}} > 0$ the particle is rotating in the same direction as the BH, and we see that it adds to the energy at infinity.

If $\mathrm{V}^{\hat{\phi}} < 0$ the particle rotates in the opposite direction as the BH, now, it subtracts from the energy at infinity!

However, as long as $E^{\text{Kerr}} > 0$ the particle will always add mass to the BH.

Negative energy

For a Kerr BH, there are two terms as we have discussed, therefore it is now possible to have negative energy!

$$E^{\mathrm{Kerr}} = -\mathrm{p}_t = \gamma m_0 \mathrm{c}^2 \frac{\rho\sqrt{\Delta}}{\Sigma} + \gamma m_0 \mathrm{V}^{\hat{\phi}} \frac{\Sigma \sin\theta}{\rho}\omega$$

We find that to achieve this, the condition is $\frac{V^{\hat{\phi}}}{c} > \frac{\rho^2\sqrt{\Delta}}{\Sigma^2 \sin\theta}\frac{c}{\omega} = \frac{\rho^2\sqrt{\Delta}}{2\mathrm{j}\mathrm{r}_g{}^2 r \sin\theta}$

Energy is now negative and this particle will actually decrease the BH mass as it falls in! This also means that in the process, it releases part of the BH mass as energy to the outside world!

However, we find that since $V^{\hat{\phi}} < c$ always, the only place where this condition can hold is within the ergosphere.

This process of extracting rotational mass-energy from the BH is called the Penrose Process.

Reducible and irreducible mass

7.5.1.2 Reducible and Irreducible Mass

If rotational energy were to be extracted from a spinning black hole, it not only would spin down, it also would lose the gravitational mass of that lost spin energy. When this process is complete, and the hole is left in a non-spinning state, the mass remaining is its "irreducible" mass[11]

$$M_{\rm ir} = M \left[\left(1 + \sqrt{1 - j^2} \right) / \, 2 \right]^{1/2} \tag{7.50}$$

which is proportional to the square root of the horizon area $4\pi(r_{\rm H}^2 + j^2 r_g^2)$. The total available rotational energy that can be extracted, therefore, is $M_{\rm red} c^2$, where $M_{\rm red} \equiv M - M_{\rm ir}$ is the amount of "reducible" mass in the spinning hole. Note that $M_{\rm red}$ is not the same as the mass that can be formed from the (angular) momentum of the spinning black hole: $M_P = Jc/(2\,GM_{\rm ir})$. The latter adds in quadrature with $M_{\rm ir}$ to give $M^2 = M_P^2 + M_{\rm ir}^2$, similar to equation (6.64).

Free fall

$$L = \gamma m_0 V^{\hat{\phi}} \frac{\Sigma \sin\theta}{\rho}$$

The free fall case is easy to compute, since it will have $V^{\hat{\phi}} = 0$ always.

This is in fact very interesting because $V^{\hat{\phi}}$ is relative to the rotating space around the Kerr BH! So, the falling body actually picks up angular velocity that is exactly that of the rotating space!

$$v^{\phi} = \frac{U^{\hat{\phi}}}{U^{\hat{t}}} = \frac{\frac{\gamma V^{\hat{\phi}} \rho}{\Sigma \sin\theta} + \frac{\gamma \Sigma \omega}{\rho\sqrt{\Delta}}}{\gamma \Sigma / \rho\sqrt{\Delta}} = \omega!$$

v^{ϕ} is the velocity measured by an observer using the BL frame.

Free fall

Now, let's discuss the detailed velocity as a function of r as the particle falls from infinity.

By setting the energy at infinity to the rest mass-energy as we did last week,

$$E(r) = m_0\left(\gamma c^2 \frac{\rho\sqrt{\Delta}}{\Sigma} + \cancel{\frac{\gamma V^{\hat{\phi}}\Sigma\sin\theta}{\rho}\omega}\right) = E_\infty = m_0 c^2$$

We find

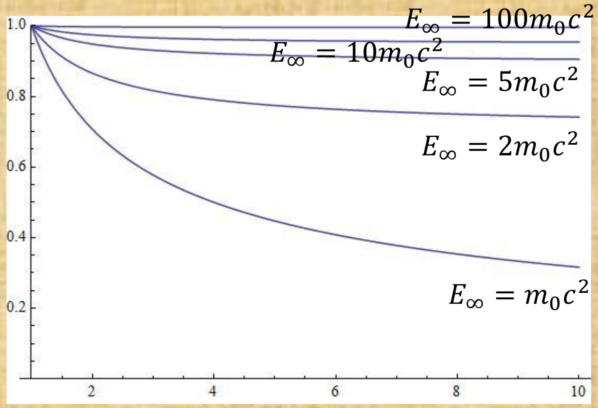
$$V^{\hat{r}}_{\mathrm{ff,BL}}(r,j) = -c\sqrt{\frac{2r_g}{r}}\left(\frac{r\sqrt{r^2+j^2{r_g}^2}}{\Sigma}\right)$$

$$\Sigma \equiv \sqrt{\left(r^2+j^2{r_g}^2\right)^2 - j^2{r_g}^2\Delta\sin^2\theta}$$

$$\Delta \equiv r^2 - 2r_g r + {r_g}^2 j^2$$

Schwarzschild case:

$$V^{\hat{r}}{}_{\mathrm{ff},SH}(r) = -c\sqrt{\frac{2r_g}{r}}$$



For the Kerr case , even a simple free-fall becomes very interesting, it depends on not only the spin but also the angle of entry θ.

Let's plot some of these on the next page and see what happens.

Free fall

Mathematica file

Orbits in the equatorial plane

Following a similar procedure as the Schwarzschild case, but much more complicated (I don't know exactly how complicated it is because I didn't do it…), we get

$$\frac{V^{\hat{\phi}}_{\mathrm{orb,BL}}}{c} = \frac{\Sigma^2}{\sqrt{\Delta}\left(r^3 - j^2 {r_g}^3\right)}\left(\pm\sqrt{\frac{r_g}{r}} - j\,\frac{{r_g}^2}{r^2}\right) - \frac{2j{r_g}^2}{r\sqrt{\Delta}}$$

Schwarzschild case:

$$\frac{V^{\hat{\phi}}_{\mathrm{orb,SH}}}{c} = \sqrt{\frac{r_g}{r - 2r_g}}$$

$\pm\sqrt{\frac{r_g}{r}}$ Keplerian rotation in the rotating space of the BH.
$+$ is prograde (in the global coordinate)
$-$ is retrograde (in the global coordinate)

$-j\,\frac{{r_g}^2}{r^2}$ Comes from the outward contrifugal effect due to rotation of space.

$-\frac{2j{r_g}^2}{r\sqrt{\Delta}}$ Because we measure $V^{\hat{\phi}}_{\mathrm{orb,BL}}$ in the rotating frame, so the velocity of the metric must be subtracted off.

A note on prograde and retrograde

$j = \frac{J}{GM^2/c}$ is the dimensionless angular momentum parameter ($-1 \le j \le 1$) that measures how much angular momentum J the BH has relative to its maximum value.

$\pm\sqrt{\frac{r_g}{r}}$

Keplerian rotation in the rotating space of the BH.
$+$ is prograde (in the global coordinate)
$-$ is retrograde (in the global coordinate)

[9] We have a few important notes on the j parameter. First, there is little consistency in the literature and books for what to call the dimensionless angular momentum. Some define an $a \equiv J/(Mc)$, which has units of length, and then a dimensionless $a^* \equiv a/r_g$. Others drop the asterisk on the latter parameter and use a for the dimensionless quantity, thereby conflicting with others in the literature. Here we use j throughout the book (and in papers we publish) as the dimensionless expression of J, just as m is for M, $\dot{m}$ is for $\dot{M}$, *etc*. Second, most authors take the range of j to be $0 \le j \le 1$ and then treat orbits of individual particles as being prograde or retrograde. However, in this book we find it both mathematically, and astrophysically, useful to envision the black hole as being at the center of a stationary (*e.g.*, Boyer–Lindquist) coordinate system in which many other objects (stars, gas, *etc.*) are moving. In that case the black hole can be rotating in a prograde ($0 \le j \le 1$) or retrograde ($-1 \le j \le 0$) fashion in that system, and orbits *still* can be prograde or retrograde also within that system. Therefore, here we allow j to have the full range from -1 to $+1$. This also still handles all four combinations of ($j < 0$, $j > 0$) and ($\Omega_{\rm orb} < 0$, $\Omega_{\rm orb} > 0$).

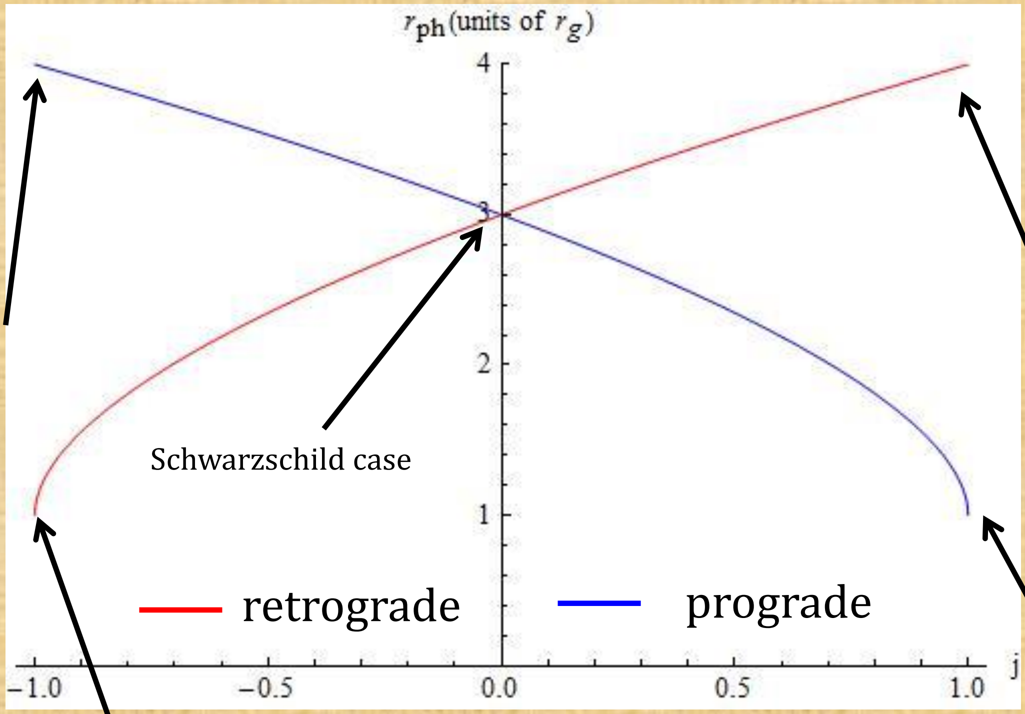
The photon orbit

Using $V^{\hat{\phi}}_{\text{orb,BL}} = c$ as last week, $r_{\text{ph,BL}} = 2r_g\left(1 + \cos\left(\frac{2}{3}\cos^{-1}(\mp j)\right)\right)$

Orbiting counter to BH spin.

Prograde orbit in j=-1 BH is further from the BH ($4\ r_g$) because it is going against the BH.

Retrograde orbit in j=1 BH is further from the BH ($4\ r_g$) because it is going against the BH.



Retrograde orbit in j=-1 BH gets close to the BH ($1\ r_g$) because it is also going in the same direction as the BH.

Prograde orbit in j=1 BH gets close to the BH ($1\ r_g$) because it is going in the same direction as the BH.

Orbiting in same direction as BH spin.

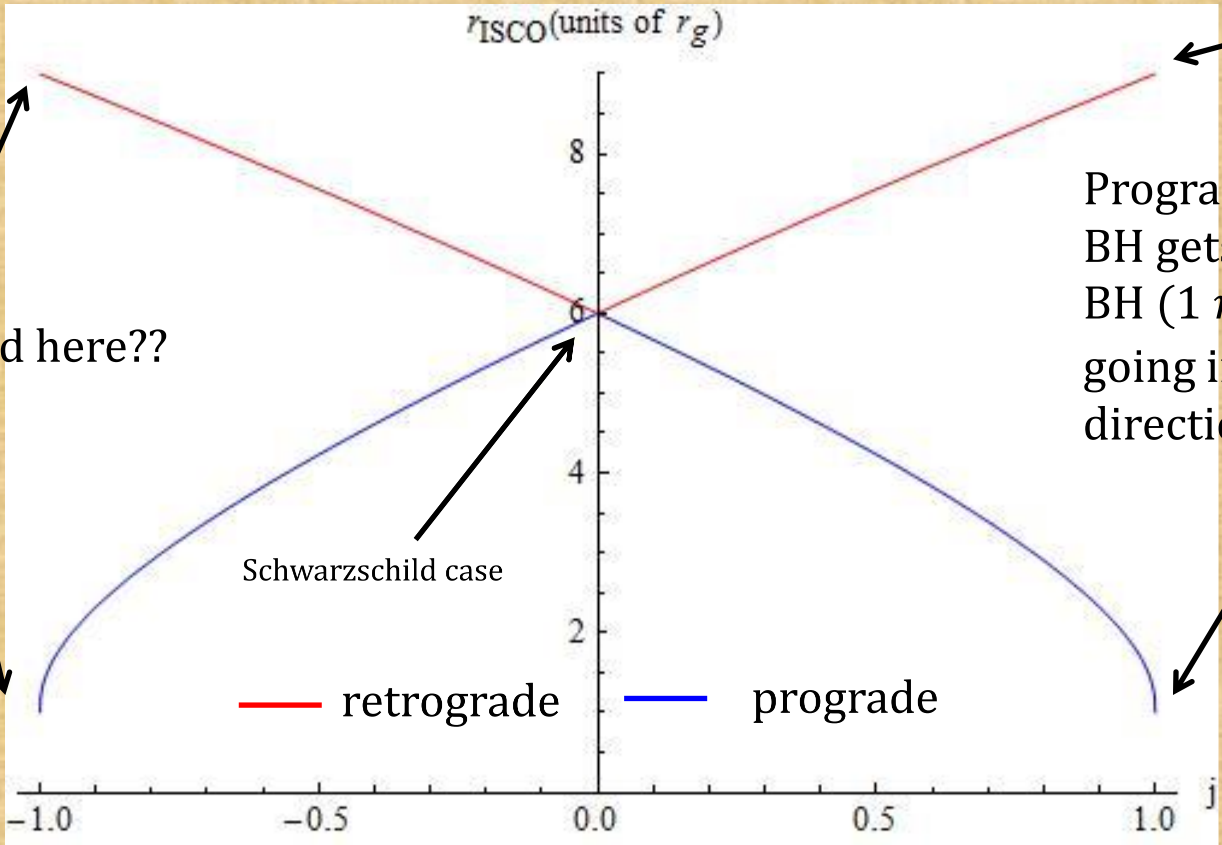
The ISCO

Following similar but more complicated derivation as last week (again, I was lazy),

$$r_{\mathrm{ISCO,BL}} = h \mp \sqrt{\frac{-h^3 + 9h^2 - 3(6-j^2)\mathrm{h} + 7j^2}{h-3}}$$

$$h = 3 + \sqrt{3 + j^2 + (1-j^2)^{\frac{1}{3}}\left((3-j)(1+j)^{\frac{1}{3}} + (3+j)(1-j)^{\frac{1}{3}}\right)}$$

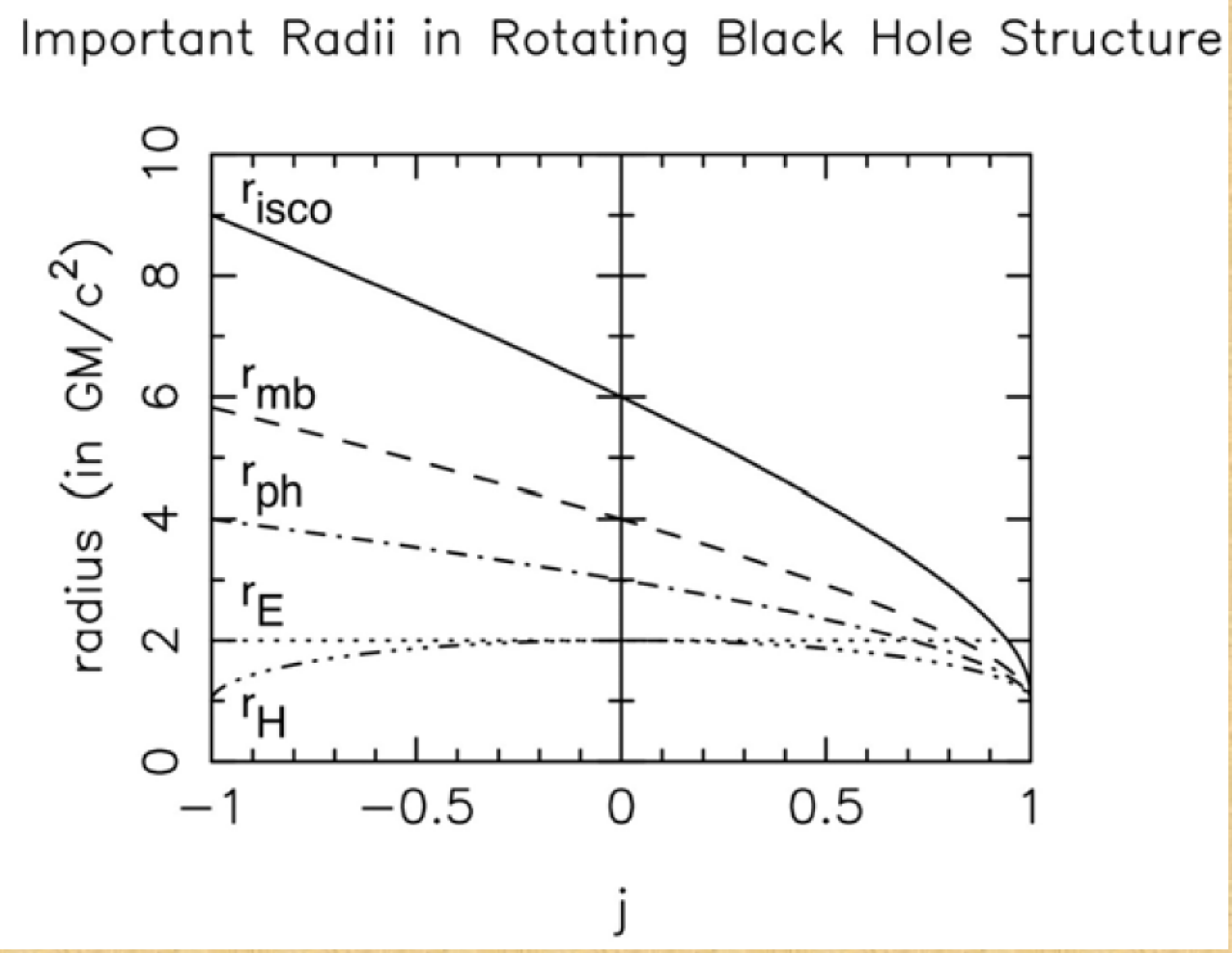
Retrograde orbit in j=1 BH is further from the BH (9 r_g) because it is going against the BH.



What happened here??

Prograde orbit in j=1 BH gets close to the BH (1 r_g) because it is going in the same direction as the BH.

Prograde orbits



Horizon Penetrating Coordinates

As can be seen from the BL coordinates, the singularity in g_{rr} comes with $\Delta \to 0$. To avoid this, we can again find a horizon penetrating coordinate similar to what we have done in the Schwarzschild BH case.

$$\left(g_{\mathrm{BL}}^{\mathrm{KER}}\right)_{\alpha\beta} = \begin{pmatrix} -\left(1-\frac{2r_g r}{\rho^2}\right)c^2 & 0 & 0 & -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} & 0 & 0 & \frac{\Sigma^2}{\rho^2}\sin^2\theta \end{pmatrix}$$

This is achieved by redefining both t and ϕ $\quad \mathrm{dt} = \mathrm{dt}' - \frac{2r_g r}{\mathrm{c}\Delta}\mathrm{dr} \qquad \mathrm{d}\phi = \mathrm{d}\phi' - \frac{\mathrm{j}r_g}{\Delta}\mathrm{dr}$

The coordinate change compresses time near the BH.

Then the new metric reads

$$\left(g_{\mathrm{HP}}^{\mathrm{KER}}\right)_{\alpha\beta} \equiv \begin{pmatrix} -\left(1-\frac{2r_g r}{\rho^2}\right)c^2 & \frac{2r_g r}{\rho^2}c & 0 & -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} \\ \frac{2r_g r}{\rho^2}c & 1+\frac{2r_g r}{\rho^2} & 0 & -\mathrm{j}r_g\sin^2\theta\left(1+\frac{2r_g r}{\rho^2}\right) \\ 0 & 0 & \rho^2 & 0 \\ -\frac{\omega\Sigma^2\sin^2\theta}{\rho^2} & -\mathrm{j}r_g\sin^2\theta\left(1+\frac{2r_g r}{\rho^2}\right) & 0 & \frac{\Sigma^2}{\rho^2}\sin^2\theta \end{pmatrix}$$